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CRITICAL CONDITIONS FOR THE GROUP SPEED OF PROPAGATION OF INTERNAL GRAVITY WAVES

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Four experimentally observed unstable and resonant regimes of generation of internal waves by a moving or oscillating cylinder are considered. Two of them can be treated as a manifestation of the critical-layer effect, but for the group rather than for the phase speed of propagation of small perturbations, one regime can be regarded as a manifestation of the effect of compaction of the energy of two waves, and one more regime admits both of the indicated treatments.

We consider two-dimensional steady flow with specified distributions of the velocity u and density ρ along the z coordinate directed vertically upward. Additional explanations are given in Fig. 1. In the unperturbed state, the stability condition $d\rho/dz \leqslant 0$ is satisfied over the entire region occupied by the liquid. A two-layer quiescent liquid is considered as a particular case ($u = 0$ and ρ varies jumpwise). A two-dimensional perturbation is introduced into this system by a cylinder of diameter D moving under the law

$$x_* = x_0 - Ut, \qquad z_* = h + a\sin(\Omega t + \varphi_0),$$

where x_* and z_* are the coordinates of the cylinder axis in the fixed system shown in Fig. 1 and x_0, U, h, a, Ω, and φ_0 are parameters. The purely translational ($\Omega = 0$) and purely oscillatory ($U = 0$) laws of motion are considered as particular cases.

The problem contains a number of characteristic speeds. For linear waves, besides explicitly prescribed values of u and U, of significance are the phase c and group c_g speeds of propagation of small harmonic perturbations. It is important that for gravity waves in a liquid, c and c_g do not coincide as a rule. For nonlinear perturbations, the notion of the group speed loses sense but the limiting speed of propagation of solitary waves $c_s^{\max}$ plays an important role.

One of the most informative indications of the critical state is the equality of some characteristic speeds. The present work focuses on the conditions $c_g = u$ and $c_g = U$. For comparison, we also give information on the critical states with satisfaction of the conditions $c = u$, $c_g = c$, and $c_s^{\max} = u$.

Results of analysis of the system response to a small perturbation with $c = u$ have given an impetus for the development of the linear theory of hydrodynamic stability. In this theory, the main flow is subjected to a small perturbation whose stream function has the form

$$\psi = \Psi(z)\exp[i(kx - \omega t)], \qquad i = \sqrt{-1}, \tag{1}$$

where x is the longitudinal coordinate, t is time, k is the wavenumber, and ω is the angular frequency. For perturbation (1), the phase and group speeds are defined by the relations $c = \omega/k$ and $c_g = d\omega/dk$.

Rayleigh [1], analyzing the case an inviscid liquid of homogeneous density ($\rho = \text{const}$) in a linear approximation, obtained the following equation for the perturbation amplitude:

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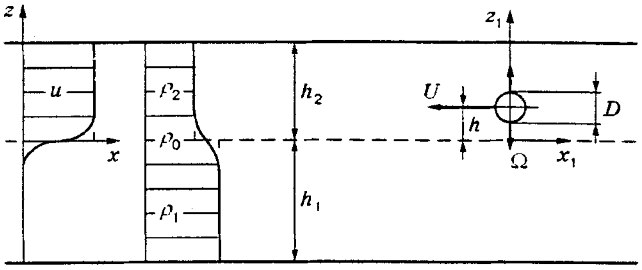

Fig. 1. The scheme of the problem.

$$(u-c)\Big(\frac{d^2\Psi}{dz^2}-k^2\Psi\Big)-\frac{d^2u}{dz^2}\Psi=0.$$

For $u=c$, it has only the trivial solution $\Psi=0$. The values of z for which $u=c$ are called the critical layer (or critical level) [2], in the given case, for the phase speed. Later, the uncertainty for $u=c$ was eliminated by two methods: taking into account nonlinearity and taking into account the real physical properties of liquids.

Taking into account the real physical properties of liquids, Orr [3] and Sommerfeld [4] obtained the equation

$$(u-c)\Big(\frac{d^2\Psi}{dz^2}-k^2\Psi\Big)-\frac{d^2u}{dz^2}\Psi+\frac{i\nu}{k}\Big(\frac{d^4\Psi}{dz^4}-2k^2\frac{d^2\Psi}{dz^2}+k^4\Psi\Big)=0,$$

where ν is the kinematic coefficient of viscosity. For $\rho=\text{const}$, it solves the problem of the critical layer in the phase speed.

For an inviscid, density stratified liquid, Taylor [5] and Goldstein [6] obtained the equation

$$(u-c)^2\Big(\frac{d^2\Psi}{dz^2}-k^2\Psi\Big)-\Big[(u-c)\frac{d^2u}{dz^2}-N^2\Big]\Psi=0,\quad N^2=-\frac{g}{\rho}\frac{d\rho}{dz} \tag{2}$$

(g is the free-fall acceleration), which has a quadratic singularity for $u=c$.

Drazin [7] took into account liquid viscosity and obtained the equation

$$(u-c)^2\Big(\frac{d^2\Psi}{dz^2}-k^2\Psi\Big)-\Big[(u-c)\frac{d^2u}{dz^2}-N^2\Big]\Psi+\frac{i\nu}{k}(u-c)\Big(\frac{d^4\Psi}{dz^4}-2k^2\frac{d^2\Psi}{dz^2}+k^4\Psi\Big)=0.$$

But this did not resolve the uncertainty for $u=c$. Only additional allowance for molecular diffusion of the substance (heat or impurity) that produces density stratification made it possible to solve the problem.

Hazel [8] derived the equation

$$L(\Psi)=0. \tag{3}$$

Here

$$L=(u-c)^2(d^2-k^2)-[u-c+a(d^2-k^2)]\frac{d^2u}{dz^2}+N^2+ab(d^2-k^2)^3-(u-c)(a+b)(d^2-k^2)^2,$$

where $d=d/dz$, $a=i\chi/k$, and $b=i\nu/k$ (χ is the molecular-diffusion coefficient). All previous equations are particular cases of (3).

The effect of molecular diffusion on the stability of shear flow of a stratified liquid was studied experimentally in [9], where a review of papers containing numerical calculations is given.

The equality of the coefficients $(u-c_g)^2$ and $u-c_g$ to zero corresponds to the critical layer for the group speed. These coefficients appear in the second approximation in a series expansion in the perturbation amplitude. Liu and Benney [10] obtained the equation

$$\Big[(u-c_g)^2\frac{d^2}{dz^2}-(u-c_g)\frac{d^2u}{dz^2}+N^2\Big]\Psi_1=\Psi^2f_0, \tag{4}$$

where Ψ is defined by Eq. (2), Ψ_1 is the amplitude of the stream function of the drift flow generated by a weakly nonlinear wave, and f_0 is a certain function of z, u, N, c, and $c - c_g$, for which an explicit expression is obtained in [10]. This expression is very cumbersome and is not given herein. We only note that for $u = c_g$,

$$\Psi_1 = \Psi^2 \frac{c - c_g}{(u - c)^2}.$$

Physical processes in the critical layers for the group speed have not been studied experimentally. The present investigation partly fills this gap. Below, we give some results from three series of experiments in a tank with a flat horizontal bottom 4.8 m long and 0.2 m wide (Fig. 1).

In the series of experiments I ($\Omega = 0$) and II ($U = 0$), the lower layer (a weak solution of glycerin in water) was at rest, and the upper layer (distilled water) moved at velocity u. Because of molecular viscosity and molecular diffusion, interlayers with small (about 1 cm) characteristic thicknesses δ_1 and δ_2 for density and velocity, respectively, formed between water and the solution. Up to a certain value of the difference in velocity between the layers, these interlayers stabilized the flow (see, e.g., [2]), and in the examples considered, the unperturbed state of the system was stable. The constant value of the velocity u and the relatively low level of turbulent fluctuations (root-mean-square value smaller than $0.02u$) was ensured by special devices located at the entrance to the working section of the tank. The surface on which $\rho = \rho_0 = (\rho_1 + \rho_2)/2$ was assumed as the conditional interface between the layers.

In the series of experiments III, the case of a two-layer quiescent liquid in the unperturbed state ($\delta_1 = \delta_2 = 0$ and $u = 0$) was studied. Stratification was produced by means of water and kerosene, and the stability of the unperturbed state against uncontrolled perturbations was ensured by interface tension.

The densities of the liquids were as follows: (0.999 ± 0.001) g/cm^3 for water, (0.8 ± 0.001) g/cm^3 for kerosene, and (1.013 ± 0.002) g/cm^3 for the glycerin solution. The kinematic-viscosity coefficients were (0.0105 ± 0.0004) cm^2/sec for water, (0.0108 ± 0.0005) cm^2/sec for the glycerin solution, and (0.0170 ± 0.0005) cm^2/sec for kerosene. The molecular-diffusion coefficient for glycerin in water was approximately $0.4 \cdot 10^{-5}$ cm^2/sec. The coefficient of interfacial tension between water and kerosene was $(40 \pm 4) \cdot 10^{-3}$ N/m. The free surface served as the upper bound, but the perturbation parameters are such that in mathematical models it can be replaced by a rigid boundary. This was established on the basis of the theoretical and experimental data of [11] and confirmed by special control experiments.

The cylinder was fastened on a towing carriage by means of two telescopic holders, whose part immersed in the liquid had a diameter of 3 mm. The gaps between the ends of the cylinder and the lateral walls of the tank were about 1 mm. Vertical oscillations of the cylinder were performed by a special device, which was able to change the amplitude a and frequency Ω of oscillations. The strictly sinusoidal nature of oscillations was provided for by a special link device.

Motion of the cylinder began from the state of rest, so that, generally, the set of specified parameters included the characteristic times of attainment of steady regimes of translational and oscillatory motions. For the examples considered below, these parameters had values of about 0.2 sec for total times of motion 30–120 sec, so that their role was insignificant. From preliminary experiments, we obtained the initial position of the cylinder x_0 relative to the right-hand end wall of the tank (Fig. 1) for which the effect of this parameter could be ignored. Observations of the processes were terminated as soon as the internal waves reflected from the end walls of the tank arrived at the x coordinate considered.

In the experiments with water and kerosene, the deviations of the interface from the equilibrium position η were measured by the wavemeters described in [11]. In the experiments with a continuously stratified liquid, photography was a more informative method. In photography, the lower layer (or the line of equal density $\rho = \rho_0 = \text{const}$) was colored by ink. The main characteristic scales of length and time were h_2 and $\sqrt{h_2/(\varepsilon g)}$, respectively ($\varepsilon = (\rho_1/\rho_2) - 1$). The fixed (x, z) and moving (x_1, z_1) coordinate systems used are shown in Fig. 1.

Planning of the experiments was performed on the basis of the semigraphical method, whose essence is explained using the example of shear flow of an inviscid two-layer liquid with no interfacial tension and with the free surface replaced by a rigid boundary (Fig. 2).

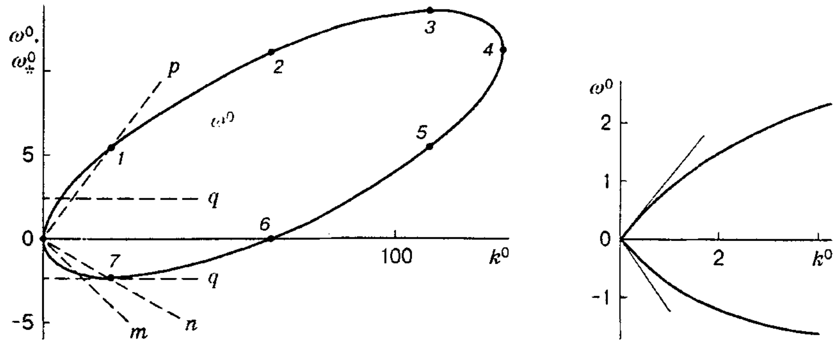

Fig. 2. Dispersion curve of the linear theory and perturbation characteristics for series of experiments I and II.

On the phase plane ($k^0 = kh_2$, $\omega^0 = \omega\sqrt{h_2/(\varepsilon g)}$), we plotted the dispersion relation of the linear theory $\omega^0(k^0)$, which in our case has the form [11]

$$\omega^0 = k^0\left[\sqrt{f} \pm \sqrt{\frac{\tanh k^0}{k^0(A+1)} - Af}\right], \tag{5}$$

where $f = F/(1+A)^2$, $F = u^2/(\varepsilon g h_2)$, $A = [(1+\varepsilon)\tanh k^0]/\tanh(k^0 H^0)$, and $H^0 = h_1/h_2$; the coordinate system is attached to the lower layer. For a quiescent unperturbed liquid, in (5) it is necessary to set $F = 0$. Formula (5) contains three independent parameters ε, F, and H^0. For $\varepsilon \ll 1$, the role of the parameter ε is minor and only the product εg is of significance. The plot in Fig. 2 is constructed for $h_2 = 18.5$ cm, $\varepsilon = 0.013$, $F = 0.031$ ($u = 27$ cm/sec), and $H^0 = 1.24$. The region of small values of k^0 is scaled up as a separate fragment.

Eight symmetrically arranged singular points for the phase and group speeds of perturbation propagation are distinguished in Fig. 2. At the point (0, 0), the phase speed assumes an extreme value, and one of the most critical conditions holds:

$$c_g^0 \to c^0 \to c_m^0 = \sqrt{f} \pm \sqrt{\frac{1}{A+1} - Af} \quad \text{for} \quad k^0 \to 0.$$

For $c^0 > c_m^0$, the linear theory is inapplicable. As a first approximation of shallow-water theory, there exist only discontinuous solutions, which in practice correspond to breaking waves. However, the experiments of [12, 13] show that shallow-water wave breaking occurs only when the speed of propagation of their leading edge $c_l > c_s^{\max} > c_m$, and in the range $c_m < c_l < c_s^{\max}$, smooth waves of the type of undular, cnoidal, etc., waves arise. The second approximation of shallow-water theory reflects this fact fairly well [11–13]. From a physical viewpoint, breaking of nonlinear waves on shallow water begins at $u = c_l$.

Critical conditions also hold at point 4, where the group speed turns to infinity. This point corresponds to the lower boundary of wavenumbers for perturbations that are unstable by the Kelvin–Helmholtz mechanism [14].

At point 2, the phase speed is equal to the rate of motion of the upper layer, and at point 6, it is equal to the rate of motion of the lower layer, i.e., the conditions determining a critical layer in the phase speed hold. At points 1, 3, 5, and 7, the conditions determining the critical layer for the group speed are satisfied. In addition, points 3 and 7 bound the existence domain of linear harmonic waves for the parameter ω^0. Between points 3 and 4 and between points 6 and 7, the speeds c_g^0 and c^0 have opposite signs. On the arc 4–5–6, $c_g^0 > c^0$ and, according to [14, 15], "negative energy" waves exist there.

On the plane (k^0, ω^0), along with $\omega^0(k^0)$, we plotted the characteristic of the introduced perturbation $\omega_*^0(k^0)$. For joint translational and oscillatory motion of the cylinder, which took place in the series of experiments III, the perturbation characteristic is represented by three parallel straight lines:

$$\omega_{*1}^0 = k^0 U^0, \quad \omega_{*2}^0 = k^0 U^0 + \Omega^0, \quad \omega_{*3}^0 = k^0 U^0 - \Omega^0, \tag{6}$$

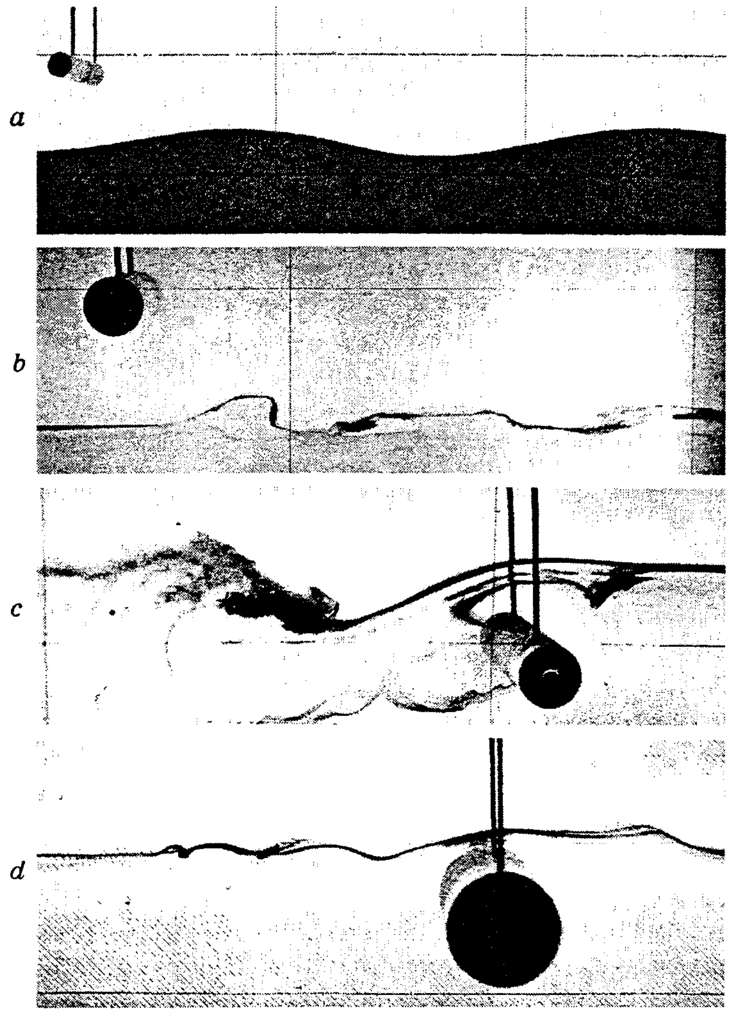

Fig. 3. Examples of physical processes in shear flow: (a), (b), (c), and (d) correspond to the perturbation characteristics m, n, p, and q in Fig. 2.

where $U^0 = U/\sqrt{\varepsilon g h}$ and $\Omega^0 = \Omega\sqrt{h_2/(\varepsilon g)}$. From the points of intersection $\omega^0(k^0)$ and $\omega_*^0(k^0)$, it is possible to determine the length, frequency, and phase and group speeds of propagation of the stationary linear harmonic waves generated by the specified perturbation. With variation in the parameters of the system and perturbation, the straight lines (6) can have up to five points of intersection with curve (5). Of special interest are combinations of parameters for which the straight lines (6) intersect or are tangent to curve (5) at some of the singular points indicated above. Below, we discuss precisely these examples.

In the particular case of purely translational motion (series of experiments I), $\Omega = 0$ and three straight lines (6) merge into one: $\omega_{*1} = \omega_{*2} = \omega_{*3} = k^0U^0$. For purely oscillatory motion (series of experiments II), $U = 0$ and the perturbation characteristic is represented by two straight lines: $\omega_{*2,3} = \pm\Omega^0$.

Relation (5) corresponds to an idealized system. In the region of large k^0, the viscosity, molecular diffusion, and interfacial tension have a significant effect on $\omega^0(k^0)$. In the present work, we calculated dispersion relations taking into account these factors. A comparison showed that for small k^0 (from $k^0 = 0$ to $k^0 \simeq 12$), formula (5) gave good accuracy, and the parameters F, h/h_2, D/h_2, a/h_2, and $\delta_{1,2}/h_2$ in the experiments were specified precisely for these values of k^0.

The straight lines m, n, and p in Fig. 2 are the perturbation characteristics from the series of experiments I. For the characteristic m, the speed of the cylinder $U^0 = -0.305$. The point of its intersection with

$\omega^0(k^0)$ is not a singular point, and the cylinder generates a smooth, almost sinusoidal wave. It is shown in the photo in Fig. 3a. The length, frequency, and phase and group speeds of this wave are adequately described by linear theory, and the amplitude is about 30% larger than the amplitude obtained in calculations with the cylinder simulated by a dipole. The causes of this difference are discussed in [11]. The cylinder moves from right to left in the upper layer at a distance from the conditional interface of $h/D = 3$, where $D = 2$ cm. The upper layer moves to the right, and the lower layer, colored by ink, is at rest.

The characteristic n corresponds to a speed of the cylinder of $U^0 = -0.126$, $D = 3$ cm, and $h/D = 3$; the remaining parameters are same as in Fig. 3a. The absolute value of the speed of the cylinder is lower than that in the previous example, and, at first glance, the waves should be even more stable. However, in this case, the point of intersection of the characteristics of the system and the perturbation are among the singular points for the group speed, and the waves turn out to be unstable (Fig. 3b).

The characteristic p corresponds to a speed of the cylinder of $U^0 = 0.305$; the point of intersection is also singular, and the perturbation is also unstable (Fig. 3c). A cylinder of 3-cm diameter moves in the lower layer in the same direction as the upper layer but slightly ahead of it. It should be noted that in the above examples, the entire upper layer or the entire lower layer was in the critical state for the group speed of perturbation propagation.

The characteristic q refers to experiments of series II with purely oscillatory motion of the cylinder. Its lower branch is tangent to the dispersion curve at the singular point 7. The system response to this perturbation is shown in Fig. 3d ($D = 7.5$ cm, oscillation amplitude $a/D = 0.5$, and $h/D = -1.25$). Although the diameter of the cylinder is much greater than that in series I, the cylinder generates weak shapeless waves, which cannot propagate far upstream and break.

We note that by the terminology of [14], in series II the regime of compaction of the energy of two waves occurs when the two points of intersection of $\omega(k)$ and $\omega_*(k)$ merge. The fact that no strengthening of waves occurred in this case can be explained by particular relations between the phases of the individual harmonic components of this perturbation. A well-known similar situation arises for the particular range of speeds of gravity and capillary waves at which they suppress each other [16].

In the absence of a velocity shear between the layers, two critical conditions for the group speed are possible. One of them is the same as in the presence of a velocity shear between the layers: $c_g = c$ as $k \to 0$. The other condition $c_g = U$ holds, for example, in translational-oscillatory motion of a cylinder. Within the framework of the model of an inviscid boundless two-layer liquid ignoring interfacial tension, this condition holds if the parameters of the law of motion (6) satisfy the relation [17]

$$\frac{(1+\varepsilon)U\Omega}{\varepsilon g} = \frac{1}{4}. \tag{7}$$

Varying values of U, Ω, and ε so as to satisfy condition (7), it is possible to obtain a set of critical regimes for the group speed. A number of such regimes occurred in the series of experiments III. It should be noted that viscosity and interfacial tension change condition (7), which was taken into account in planning experiments by the semigraphical method described above. The error did not exceed several percent. However, for the resonant regime, it was significant. The most pronounced physical effects were observed for combinations of parameters that were specified previously in the experimental part of [17]: $h_1 = 30$ cm, $h_2 = 15$ cm, $D = 1$ cm, $h = 3$ cm, $a = 0.5$ cm, and $\varepsilon = 0.25$. The parameters U and Ω were varied.

Figure 4 shows experimental curves of the deviation of the interface from the equilibrium position $\eta(t)$ obtained by a fixed wavemeter at $x - x_0 = -150$ cm [17]. The time t is read from the moment of start of the cylinder. The η axis intersects the t axis at the point corresponding to the moment when the cylinder axis passes above the wavemeter. The perturbations ahead of the cylinder are located to the left of the η axis.

The curve in Fig. 4a is obtained for $U = -8.44$ cm/sec and $\Omega/(2\pi) = 0.51$ Hz, and with a correction for the effect of viscosity and interfacial tension, the resonance condition is satisfied exactly. In this case, in the neighborhood of the cylinder, a stationary wave packet forms, whose envelope is reminiscent of a solitary wave. Directly under the cylinder, the oscillation period changes significantly.

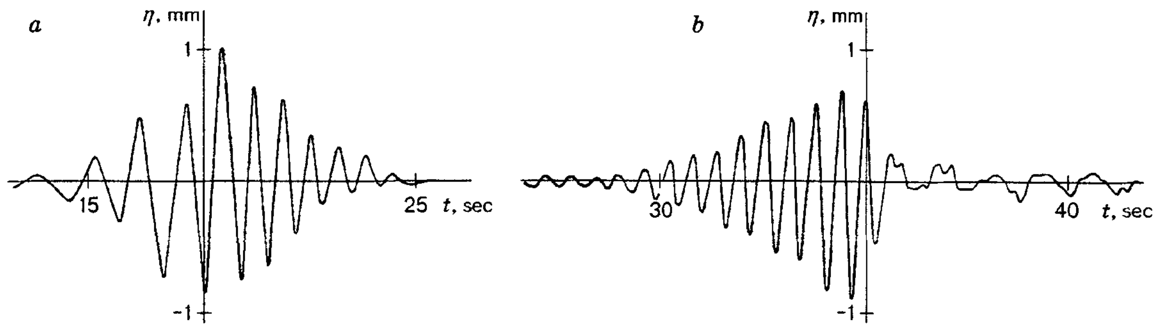

Fig. 4. Resonant (a) and nearly resonant (b) regimes in translational-oscillatory motion of a cylinder in a quiescent two-layer liquid.

This regime can also be treated as the effect of compaction of the energy of two waves. In this regime, in contrast to the regime shown in Fig. 3d, there was a manifold increase in the perturbation amplitude compared to the cases of purely translational or purely oscillatory motion of a cylinder with the same parameters.

The curve in Fig. 4b is obtained for $U = -4.28$ cm/sec and $\Omega/(2\pi) = 1.03$ Hz. With allowance for the effect of viscosity and interfacial tension, the parameters U and Ω are only 3% smaller than the critical values. The wave pattern, however, changes significantly. It is nonstationary, and multiple strengthening of perturbations occurs only ahead of the cylinder. It should be noted that in the resonant regime and in a certain neighborhood of it, the parameter φ_0 in the law of motion of the cylinder plays a very important role. When φ_0 in the resonant regime was varied, directly under the cylinder there was a ridge or a cavity or the value of η was intermediate between them. In the nearly resonant regimes, this parameter determines where strengthening of the waves occurs: ahead of or behind the cylinder.

The data given in Fig. 4 made it possible to determine which of the two methods is more effective in a theoretical analysis of the critical regimes of wave generation: allowance for physical factors in the model or allowance for nonlinearity. A comparison with the linear theory shows [17] that, ignoring viscosity, this theory is adequate for describing the phase pattern of waves but predicts unlimited increase in their amplitude. Allowance for viscosity in the linear model led to satisfactory agreement with the experiment for wave amplitudes as well.

Thus, the above examples show that the critical conditions for the group speed of perturbation propagation play an important role in the problem of the stability of gravity waves. The quantity c_g characterizes the mean rate of transfer of the energy of a harmonic perturbation. Therefore, the flow rearrangement is faster and stronger under the critical conditions for c_g than under the critical conditions for c. At the same time, from the theoretical and experimental information obtained it follows that in the critical layers for c_g, the loss of stability is likely to proceed by a rigid type, i.e., the perturbation intensity should exceed a certain threshold value. This is confirmed by two facts: 1) the uncertainty for $c_g = u$ occurs only in the nonlinear approximation for the perturbation amplitude (4); 2) for the flows shown in Fig. 3b and c, the Richardson number $\mathrm{Ri} = \varepsilon g\delta/u^2$ was much larger than the critical value $\mathrm{Ri} = 0.25$, and, according to the linear theory, the perturbation should be stable [2]. There is no doubt that for infinitesimal perturbations, this theoretical result is valid. In addition, it is confirmed experimentally for rather small real perturbations (see Fig. 3a) but not under the critical conditions for the group speed.

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